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A SIMPLE MODEL OF THE INTERPLANETARY MAGNETIC FIELD.

I - CALCULATION OF THE MAGNETIC FIELD

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A SIMPLE MODEL OF THE INTERPLANETARY MAGNETIC FIELD I:

CALCULATION OF THE MAGNETIC FIELD

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ABSTRACT

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A simple axisymmetric model of the interplanetary magnetic field is described and solved analytically. In this model, space is divided into three regions by two concentric spheres, conductivities are assumed to be isotropic and constant in each region and flow velocities are prescribed a priori. The innermost region rotates rigidly around the axis of a dipole embedded in its center, the intermediate region contains a compressible fluid (an idealization of the solar wind) flowing radially outward with constant velocity and finally, the outermost region is at rest. Special attention is given to the limiting case of infinite conductivity, to the "garden-hose effect", to the electric field and to the effects of a constant field, aligned with the central dipole, in the outermost region.

INTRODUCTION

Experimental evidence available now indicates the existence of an interplanetary field, originating in the sun and extending at least to the earth's orbit, possibly much further. There are still many uncertain points concerning this field, but two of its main general properties have been predicted theoretically and seem to be, so far, in agreement with experiment. Both may be regarded as manifestations of the fact that a highly conducting fluid--here, the solar wind emanating radially from the sun--tends to impart its motion to magnetic lines of force embedded in it. First, it was predicted that the solar wind will stretch the lines of force, rendering them almost radial and causing the field intensity B to fall off less rapidly then it would otherwise (e.g. [Alfven, 1956]). Secondly, in addition to radial stretching, the field was expected to be twisted by solar rotation into an archimedean spiral. This point was noted first by Chapman (1928) who observed that the locus of a particle stream constantly emitted from a point on the sun is, at any time, such a spiral (the same locus is described by droplets emitted from a rotating sprinkler, for which reason the above is sometimes called the "garden-hose effect"). A line of force drawn out by a stream of particles would also follow such a spiral, and it was argued that a similar twisting occurs in rotationally symmetric situations, even though the arguments presented for this (Parker, 1958; Axford, Dessler and Gottlieb, 1963) were somewhat unsatisfactory. The effect has also been deduced from experimental data, from the cosmic ray flare effect (McCracken, 1962) ~~Pioneer V data (Greenstadt, 1963)~~ and Mariner II data (Davis, 1963); the "garden-hose angle" χ between $\underline{B}$ and the radial direction from the sun in all those cases was of the order of 45°.

In this work, a simple model of the interplanetary field will be investigated, first in the limiting case of a perfectly conducting fluid and then for the case of finite, homogeneous and isotropic conductivity. The model is as follows:

Let space be divided into three regions by two concentric spheres of radii R_0 and R_1 (fig. 1). Region I, the innermost, is assumed to rotate rigidly with angular velocity ω. This region also contains the source of magnetic field, which will be assumed to be a point dipole at the origin, with dipole moment $\underline{m}$ parallel to the axis of rotation. Region II, between the spheres, contains a compressible conducting fluid flowing out radially with constant velocity u. Finally, in region III which extends to infinity, no motion takes place. Region I here represents the sun, region II the space swept by the solar wind. The equation connecting the magnetic field $\underline{B}$, the electric field $\underline{E}$, the velocity $\underline{v}$ and the conductivity σ is

$$\mathrm{curl}\ \underline{B} = \mu_0 \sigma \left(\underline{E} + [\underline{v} \times \underline{B}] \right) \tag{1}$$

and we shall be interested in stationary solutions with rotational symmetry around $\underline{m}$, with particular interest in the case when σ is large and R_1 considerably in excess of R_0.

It should be borne in mind all along that the preceding is a gross oversimplification of the actual situation. First of all, it is not even certain whether the solar dipole field plays a major role in creating the interplanetary field. Certainly, intense local fields due to active areas influence it considerably and cause it to depart from rotational symmetry. Next, the solar wind is not an ordinary conducting fluid but rather a near-collisionless plasma, the conductivity of which is highly anisotropic; its flow may also very well prove to be turbulent. Furthermore, the imposed boundary conditions assume sharp discontinuities where gradual transitions probably occur. Finally, in any problem of this sort, the velocity $\underline{v}$ is generally not determined a priori but has to be solved simultaneously with $\underline{B}$, using the hydromagnetic flow equation (e.g. [Chandrasekar, 1956]). In the vicinity of the earth, of course, the mass flow dictates the magnetic field because of its much higher energy density; nevertheless, in the vicinity of what corresponds to the outer sphere in this model, the flow will be considerably distorted by the field.

Unfortunately, a more realistic model would be very hard to solve analytically. It is hoped, however, that the results obtained here will give some qualitative insight about the behavior of the actual interplanetary field.

THE LIMITING CASE

If σ tends to infinity while $\underline{B}$ stays finite, we generally get

$$\underline{E} \rightarrow -[\underline{v} \times \underline{B}]$$

Since $\underline{B}$ is rotationally symmetric and stationary, $\partial\underline{B}/\partial t$ vanishes and one has, in the limit of infinite σ

$$\mathrm{curl}\,[\underline{v} \times \underline{B}] = 0$$

in region II

$$[\underline{v} \times \underline{B}] = u \left(\underline{i}_\varphi B_\vartheta - \underline{i}_\vartheta B_\varphi \right)$$

The φ component of the curl gives

$$B_\varphi = \zeta(\vartheta)/r \tag{2a}$$

where $\zeta(\theta)$ is an arbitrary function. The other two components give

$$B_\vartheta = A / (r \sin\vartheta)$$

This solution, however, is singular on the symmetry axis unless $A = 0$.

Therefore

$$B_\vartheta = 0 \tag{2b}$$

The condition div $\underline{B} = 0$ determines the form of the third component

as
$$B_r = \xi(\theta)/r^2 \tag{2c}$$

with $\xi(\theta)$ another arbitrary function.

On the surface $r = R_0$, E_ϑ is continuous. Just inside the boundary, $E_\vartheta = - v_\varphi B_r$ and since B_r is continuous too, we may use its value outside the boundary to give

$$E_\vartheta(R_0) = -(\omega/R_0) \sin\vartheta \; \xi(\vartheta)$$

The continuity of E_ϑ now gives

$$\zeta(\vartheta) = -(\omega/u) \sin\vartheta \; \xi(\vartheta)$$

giving as the tangent of the "garden-hose angle"

$$\operatorname{tg} \chi = B_\varphi / B_r = -(\omega/u) \, r \sin\vartheta \tag{3}$$

Several points in the above results deserve notice. First, in a source-free region, the various components of the magnetic field fall off at different rates with distance, the higher multipoles decreasing faster. For instance, the earth's magnetic field (neglecting effects of the magnetopause) approximates a dipole much better some distance away than on its surface. This is not true for the field in interplanetary space (which is not source-free): in the limit of infinite conductivity, all multipole components of the field fall away as the inverse square of the distance. It should also be noted that the field contains an undetermined function $\xi(\vartheta)$; this can only be evaluated when more information about the field's source is given. Finally, choosing the solar radius for R_o and assuming $\underline{B}$ at the solar surface is one gauss, one finds at the earth's distance (approx. 200 R_o) $B_r \cong 2.5\gamma$, in agreement with observation (Coleman, Davis and Sonett, 1960). At this distance, taking appropriate values for u and ω, B is found to be of the same order as B_r. There is thus no disagreement between the strength of the solar surface field and that of the interplanetary field near the earth.

THE CASE OF FINITE CONDUCTIVITY

We now turn to solving the problem for arbitrary σ. We shall assume σ is uniform in each region and takes the values σ_1, σ_2 and σ_3 in regions I, II and III respectively. The following theorem is found useful: if a vector field $\underline{B}$ satisfies

$$\operatorname{div} \underline{B} = 0$$

it may be decomposed uniquely in the following manner

$$\underline{B} = \operatorname{curl} \Psi_1 \underline{r} + \operatorname{curl} \operatorname{curl} \Psi_2 \underline{r} \tag{4}$$

The theorem has been proved for rotational symmetry by Lüst and Schlüter [1954] and for the general case by Backus [1958]; following Elsasser [1946] the component fields will be termed the toroidal and poloidal components, respectively. The following identities hold generally [Smythe, 1950]:

$$\operatorname{curl} \Psi \underline{r} \equiv [\operatorname{grad} \Psi \times \underline{r}] \tag{5a}$$

$$\operatorname{curl} \operatorname{curl} \Psi \underline{r} \equiv \operatorname{grad} \frac{\partial}{\partial r}(\Psi r) - \underline{r}\, \nabla^2 \Psi \tag{5b}$$

and for rotational symmetry

$$\operatorname{curl} \Psi \underline{r} = - \underline{i}_\varphi \frac{\partial \Psi}{\partial \vartheta} \tag{6a}$$

$$\operatorname{curl} \operatorname{curl} \Psi \underline{r} = - \underline{i}_r \frac{1}{r} \Lambda^2 \Psi + \underline{i}_\vartheta \frac{1}{r} \frac{\partial^2}{\partial r \partial \vartheta} (r \Psi) \tag{6b}$$

where

$$\Lambda^2 \Psi = \frac{1}{\sin\vartheta} \frac{\partial}{\partial \vartheta} \sin\vartheta \frac{\partial \Psi}{\partial \vartheta}$$

and for any Legendre polynomial $P_n(\cos\vartheta)$

$$\Lambda^2 P_n = - n(n+1) P_n$$

It should be noted that Ψ_1 and Ψ_2 are determined within an arbitrary function of r, and also that (for rotational symmetry) the toroidal component is the φ component of the field. If $\underline{B}$ satisfies

$$\nabla^2 \underline{B} = 0 \tag{7}$$

Ψ_1 and Ψ_2 may be specified to be harmonic [Smythe, 1950].

Since $\underline{B}$ is assumed to stationary, one may write

$$\underline{E} = - \operatorname{grad} \Psi_0 \tag{8}$$

The problem thus reduces to finding the three scalar functions Ψ_0, Ψ_1 and Ψ_2 by means of the three scalar equations constituting equation (1)

In region I

$$\underline{v} = [\underline{\omega} \times \underline{r}]$$

and it may be verified that

$$[\underline{v} \times \underline{B}] = - \omega \operatorname{grad} \left(r \sin\vartheta \frac{\partial \Psi_2}{\partial \vartheta} \right) \tag{9}$$

Thus, equation (7) holds and we get

$$\nabla^2 \Psi_1 = \nabla^2 \Psi_2 = 0$$

To obtain an expression for Ψ_0, we insert equations (5b), (8) and (9) into (1) and get

$$\operatorname{grad} \left\{ \frac{\partial}{\partial r}(\Psi_1 r) + \mu_0 \sigma_1 \Psi_0 + \mu_0 \sigma_1 \omega r \sin\vartheta \frac{\partial \Psi_2}{\partial \vartheta} \right\} = 0$$

or

$$\Psi_0 = - \omega r \sin\vartheta \frac{\partial \Psi_2}{\partial \vartheta} - \frac{1}{\mu_0 \sigma_1} \frac{\partial}{\partial r}(\Psi_1 r) + c_1 \tag{10}$$

It is useful to expand Ψ_1 and Ψ_2 in spherical harmonics. This expansion will have a singularity at the origin due to the dipole field, which may be derived from a poloidal potential

$$\Psi_2(\text{dipole}) = \frac{\mu_0 m}{4\pi} \frac{\cos\vartheta}{r^2} \tag{11}$$

Thus, letting $\rho = r/R_0$

$$\Psi_1 = \sum a_n \rho^n P_n(\cos\vartheta) \tag{12a}$$

$$\Psi_2 = \sum b_n \rho^n P_n(\cos\vartheta) + \frac{\mu_0 m}{4\pi R_0^2}\frac{\cos\vartheta}{\rho^2} \tag{12b}$$

<u>In region II</u> using (5b) and (6a)

$$[\underset{\sim}{v} \times \underset{\sim}{B}] = u\, \underset{-}{i}_\vartheta \frac{\partial\Psi_1}{\partial\vartheta} - u \operatorname{curl} \underset{-}{r}\left(\frac{1}{r}\frac{\partial}{\partial r}\Psi_2 r\right) \tag{13}$$

substituting this and eq. (5a), (5b) and (8) into (1) gives

$$\operatorname{grad}\frac{\partial}{\partial r}(\Psi_1 r) - \underset{-}{r}\nabla^2\Psi_1 - \operatorname{curl}\underset{-}{r}\nabla^2\Psi_2 = -\mu_0\sigma_2\left[\operatorname{grad}\Psi_0 - u\,\underset{-}{i}_\vartheta\frac{\partial\Psi_1}{\partial\vartheta} + u \operatorname{curl}\underset{-}{r}\left(\frac{1}{r}\frac{\partial}{\partial r}\Psi_2 r\right)\right] \tag{14}$$

collecting φ components gives

$$\operatorname{curl}\underset{-}{r}\left[\nabla^2\Psi_2 - \mu_0\sigma_2 u\,\frac{1}{r}\frac{\partial}{\partial r}\Psi_2 r\right] = 0 \tag{15}$$

which may be integrated to give

$$\nabla^2\Psi_2 - \mu_0\sigma_2 u\,\frac{1}{r}\frac{\partial}{\partial r}(\Psi_2 r) = h(r) \tag{16}$$

The general solution of this is the sum of the solution of the homogeneous equation and an arbitrary function of r. Since, however, Ψ_2 is defined within such an arbitrary function, we may set h(r) = 0. In terms of the dimensionless variable ρ one then obtains

$$\nabla^2\Psi_2 - \mu_0\sigma_2 u R_0\,\frac{1}{\rho}\frac{\partial}{\partial\rho}(\Psi_2\rho) = 0 \tag{17}$$

The dimensionless quantity

$$2\alpha = \mu_0\sigma_2 u R_0 \tag{18}$$

may be regarded as the magnetic Reynolds number [Elsasser, 1956] of the system. The limiting case of very high conductivity occurs when it is much larger than unity.

Next, eliminate the curl terms from (14) by means of (15) and take the curl of the result. An equation is obtained which may be integrated with respect to ϑ, yielding an equation for Ψ_1 similar to (16). For the same reason as before, the arbitrary function of r is taken as zero, giving

$$\nabla^2\Psi_1 \quad - \quad \mu_0\sigma_2 u \frac{1}{r}\frac{\partial}{\partial r}(\Psi_1 r) \quad = \quad 0 \tag{19}$$

or in dimensionless variables

$$\nabla^2\Psi_1 \quad - \quad \frac{2\alpha}{\rho}\frac{\partial}{\partial\rho}(\Psi_1\rho) \quad = \quad 0 \tag{20}$$

Finally, in equation (14) from which the curl terms have been eliminated, substitute $\nabla^2\Psi_1$ from (19). The result may be written

$$\text{grad}\left\{ \frac{\partial}{\partial r}(\Psi_1 r) - \Psi_1 r u\mu_0\sigma_2 + \Psi_0\mu_0\sigma_2 \right\} \quad = \quad 0$$

giving

$$\Psi_0 \quad = \quad u\Psi_1 r - \frac{1}{\mu_0\sigma_2}\frac{\partial}{\partial r}(\Psi_1 r) + C_2 \tag{21}$$

Let us seek a solution of (17) of the form

$$\Psi_2 \quad = \quad \sum g_n(\rho)\, P_n(\cos\theta)$$

defining $\quad y_n(\rho) = \rho\, g_n(\rho)$

the following equations are obtained

$$y_n'' \quad - \quad 2\alpha\, y_n' \quad - \quad n(n+1)\rho^{-2} y_n \quad = \quad 0 \tag{22}$$

The first derivative is eliminated by substituting

$$y_n \quad = \quad u_n(\rho)\, e^{\alpha\rho}$$

giving

$$u_n'' \quad - \quad u_n\left(\alpha^2 + n(n+1)/\rho^2\right) \quad = \quad 0 \tag{23}$$

The last equation may be integrated analytically ([Murphy, 1960]; p. 337, eq. 256), giving

$$u_n(\rho) \quad = \quad \rho^{-(n+1)}\left(\rho^3\frac{d}{d\rho}\right)^n\left\{\rho^{-(2n-1)}\left(A_1 e^{\alpha\rho} + A_2 e^{-\alpha\rho}\right)\right\} \quad . \tag{24}$$

Taking $(A_1, A_2) = (1,0), (0, [-1]^n)$ we obtain two independent solutions which lead to two independent solutions for g_n, denoted g_{n1} and g_{n2}

$$\Psi_2 \quad = \quad \sum\left[b_{n1} g_{n1}(\rho) + b_{n2} g_{n2}(\rho) \right] P_n(\cos\theta) \tag{25}$$

and similarly

$$\Psi_1 \quad = \quad \sum\left[a_{n1} g_{n1}(\rho) + a_{n2} g_{n2}(\rho) \right] P_n(\cos\theta) \tag{26}$$

In region III, equation (1) reduces to

$$\operatorname{curl} \underline{B} = \mu_0 \sigma_3 \underline{E}$$

giving
$$\nabla^2 \Psi_0 = \nabla^2 \Psi_1 = \nabla^2 \Psi_2 = 0 \tag{27}$$

using (5b) gives

$$\operatorname{grad} \left\{ \frac{\partial}{\partial r}(\Psi_1 r) + \mu_0 \sigma_3 \Psi_0 \right\} = 0$$

or

$$\Psi_0 = -\frac{1}{\mu_0 \sigma_3} \frac{\partial}{\partial r}(\Psi_1 r) + c_3 \tag{28}$$

Ψ_1 and Ψ_2 may be expanded in spherical harmonics

$$\Psi_1 = \sum a_n'' \left(\frac{\rho_1}{\rho}\right)^{n+1} P_n(\cos\theta) \tag{29a}$$

$$\Psi_2 = \sum b_n'' \left(\frac{\rho_1}{\rho}\right)^{n+1} P_n(\cos\theta) \tag{29b}$$

where $\rho_1 = R_1/R_0$

THE POLOIDAL FIELD

On the boundaries, the three components of $\underline{B}$ and the electric potential are continuous (the continuity of the normal component of curl $\underline{B}$ does not add any new condition here). In this case, where the boundaries are spherical, the quantities continuous across the two boundaries are

$$\frac{\partial \Psi_1}{\partial \theta}, \quad \Lambda^2 \Psi_2, \quad \frac{\partial^2}{\partial \rho \partial \theta}(\Psi_2 \rho), \quad \Psi_0$$

giving 8 equations for the 8 sets of unknown coefficients previously defined. Let $\beta = \mu_0 m/4\pi R_0^2$ and let dashes denote derivatives with respect to . If $n \neq 1$, the following relations hold

$$\begin{aligned} b_n &= b_{n1}\, g_{n1}(1) + b_{n2}\, g_{n2}(1) \\ n\, b_n &= b_{n1}\, g_{n1}'(1) + b_{n2}\, g_{n2}'(1) \\ b_n'' &= b_{n1}\, g_{n1}(\rho_1) + b_{n2}\, g_{n2}(\rho_1) \\ -(n+1)\, b_n'' &= b_{n1}\, \rho_1 g_{n1}'(\rho_1) + b_{n2}\, \rho_1 g_{n2}'(\rho_1) \end{aligned} \tag{30}$$

These equations are homogeneous and will in general have only the trivial solution in which all coefficients vanish. For n = 1, however

$$
\begin{aligned}
b_1 + \beta &= b_{11}\, g_{11}(1) + b_{12}\, g_{12}(1) \\
b_1 - 2\beta &= b_{11}\, g_{11}'(1) + b_{12}\, g_{12}(1) \\
b_1'' &= b_{11}\, g_{11}(\rho_1) + b_{12}\, g_{12}(\rho_1) \\
-2b_1'' &= b_{11}\, \rho_1\, g_{11}'(\rho_1) + b_{12}\, \rho_1\, g_{12}'(\rho_1)
\end{aligned}
\tag{31}
$$

From eq. (24)

$$g_{11}(\rho) = \frac{e^{2\alpha\rho}}{\rho}\left(\alpha - \frac{1}{\rho}\right) \tag{32a}$$

$$g_{12}(\rho) = \frac{1}{\rho}\left(\alpha + \frac{1}{\rho}\right) \tag{32b}$$

The solution of (31) is then readily obtained as

$$
\begin{aligned}
b_{11} &= 3\beta\delta d \\
b_{12} &= -3\beta\delta c
\end{aligned}
\tag{33}
$$

where

$$
\begin{aligned}
a &= -e^{2\alpha}(2\alpha^2 - 4\alpha + 3) \\
b &= 2\alpha + 3 \\
c &= e^{2\alpha\rho_1}(2\alpha^2 - \alpha) \\
d &= \alpha/\rho_1 \\
\delta &= (ad - bc)^{-1}
\end{aligned}
\tag{34}
$$

from this

$$b_1 = b_{11}\, e^{2\alpha}(\alpha - 1) + b_{12}(\alpha + 1) - \beta$$

$$\rho_1^2 b_1'' = b_{11}\, e^{2\alpha\rho_1}(\alpha\rho_1 - 1) + b_{12}(\alpha\rho_1 + 1)$$

It should be noted that so far ω , σ_1 and σ_3 have not entered. The results obtained reflect only the stretching of the field's radial component and have nothing to do with the rotation of region I.

It is of interest to note the behavior of the solution if both α and ρ_1 are somewhat (an order of magnitude or more) larger than unity. Then

$$\delta \cong -1/bc$$

$$b_{11} \cong -3\beta e^{-2\alpha\rho_1}/4\alpha^2\rho_1 \tag{35}$$

$$b_{12} \cong 3\beta/2\alpha$$

so in region II

$$\Psi_2 \cong \left[-3\beta e^{2\alpha(\rho-\rho_1)}/4\alpha\rho\rho_1 + 3\beta/2\rho\right] \cos\vartheta \tag{36}$$

Under the assumptions made the first term, contributed by b_{11}, is dominant everywhere except near the outer boundary, where the field departs from the radial direction and curves to meet the outside dipole field. By equation (6b)

$$B_r = 3\beta R_0 \cos\vartheta / r^2$$

$$B_\vartheta = (3\beta R_0 \sin\vartheta / 2 r R_1)\, e^{-2\alpha(\rho_1-\rho)}$$

in agreement with the limiting solution. In the limit

$$b_1 \cong \beta/2$$

from which, in region I

$$\Psi_2 = \beta\cos\vartheta\left(\frac{\rho}{2} + \frac{1}{\rho^2}\right) \tag{37}$$

The poloidal field at $\rho = 1$ is thus 3/2 times the field produced there by the central dipole alone.

THE TOROIDAL FIELD

Defining

$$2\alpha_1 = \mu_0\sigma_1 u R_0$$

$$2\alpha_3 = \mu_0\sigma_3 u R_0$$

$$\Psi_{01} = \Psi_0 / u R_0$$

equations (10), (21) and (28) become, in dimensionless units of length,

(I) $$\Psi_{01} = \frac{\omega}{u}\left(\ell_1 \rho^2 + \beta/\rho\right)\sin^2\vartheta - \frac{1}{2\alpha_1}\frac{\partial}{\partial\rho}(\Psi_1\rho) + c_1 \qquad (38)$$

(II) $$\Psi_{01} = \Psi_1\rho - \frac{1}{2\alpha}\frac{\partial}{\partial\rho}(\Psi_1\rho) + c_2 \qquad (39)$$

(III) $$\Psi_{01} = -\frac{1}{2\alpha_3}\frac{\partial}{\partial\rho}(\Psi_1\rho) + c_3 \qquad (40)$$

developing Ψ_1 as in (12a), (26) and (29a) we obtain

$$a_n = a_{n1}\, g_{n1}(1) + a_{n2}\, g_{n2}(1) \qquad (41a)$$

$$a_n'' = a_{n1}\, g_{n1}(\rho_1) + a_{n2}\, g_{n2}(\rho_1) \qquad (41b)$$

and two more sets of equations derived from the continuity of Ψ_{01}.

Since

$$\sin^2\vartheta = (2/3)\,(P_0 - P_2)$$

the set of equations obtained for any n but 2 or 0 is homogeneous and its only general solution is zero. The n = 0 term doesn't contribute to B, so we concentrate on the quadrupole term. Using eq. (24) one finds in region II, for the quadrupole term only

$$\Psi_1 = \left\{\frac{a_{21}}{\rho} e^{2\alpha\rho}\left(\alpha^2 - \frac{3\alpha}{\rho} + \frac{3}{\rho^2}\right) + \frac{a_{22}}{\rho}\left(\alpha^2 + \frac{3\alpha}{\rho} + \frac{3}{\rho^2}\right)\right\} P_2(\cos\vartheta) \qquad (42a)$$

$$\Psi_1\rho - \frac{1}{2\alpha}\frac{\partial}{\partial\rho}(\Psi_1\rho) = \left\{ a_{21}\cdot 3 e^{2\alpha\rho}\left(\frac{1}{\alpha\rho^3} - \frac{1}{2\rho^2}\right) + a_{22}\left(\alpha^2 + \frac{3\alpha}{\rho} + \frac{9}{2\rho^2} + \frac{3}{\alpha\rho^3}\right)\right\} P_2(\cos\vartheta) \qquad (42b)$$

accordingly, we define

$$\gamma = -\frac{2}{3}(\ell_1 + \beta)\frac{\omega}{u} \qquad (43)$$

$$\lambda_1 = -3\alpha_1 e^{2\alpha}\left(\frac{1}{2} - \frac{1}{\alpha}\right)$$

$$\lambda_2 = \alpha_1\left(\alpha^2 + 3\alpha + 9/2 + 3/\alpha\right)$$

$$\lambda_3 = -\frac{3\alpha_3}{2\rho_1^2} e^{2\alpha\rho_1}\left(1 - 2/\alpha\rho_1\right)$$

$$\lambda_4 = \alpha_3\left(\alpha^2 + 3\alpha/\rho_1 + 9/2\rho_1^2 + 3/\alpha\rho_1^3\right)$$

$$\lambda_5 = e^{2\alpha}\left(\alpha^2 - 3\alpha + 3\right)$$

$$\lambda_6 = \alpha^2 + 3\alpha + 3$$

$$\lambda_7 = \frac{1}{\rho_1} e^{2\alpha\rho_1}\left(\alpha^2 - \frac{3\alpha}{\rho_1} + \frac{3}{\rho_1^2}\right)$$

$$\lambda_8 = \frac{1}{\rho_1}\left(\alpha^2 + \frac{3\alpha}{\rho_1} + \frac{3}{\rho_1^2}\right)$$

$$\Delta = \left[\left(\lambda_1 + \tfrac{3}{2}\lambda_5\right)(\lambda_4 - \lambda_8) - \left(\lambda_2 + \tfrac{3}{2}\lambda_6\right)(\lambda_3 - \lambda_7)\right]^{-1}$$

Then the boundary conditions give

$$\alpha_1 \gamma - \tfrac{3}{2} a_2 = \lambda_1 a_{21} + \lambda_2 a_{22} \tag{44}$$

$$a_2'' = \lambda_3 a_{21} + \lambda_4 a_{22}$$

$$a_2 = \lambda_5 a_{21} + \lambda_6 a_{22}$$

$$a_2'' = \lambda_7 a_{21} + \lambda_8 a_{22}$$

from which

$$a_{21} = \alpha_1 \gamma \Delta (\lambda_4 - \lambda_8) \tag{45}$$

$$a_{22} = -\alpha_1 \gamma \Delta (\lambda_3 - \lambda_7)$$

suppose $\alpha \gg 1$ $\rho_1 \gg 1$. Then the order of magnitudes of the coefficients is as follows

$$\lambda_7 > (-\lambda_3) > \lambda_5 > (-\lambda_1) > \lambda_4 \cong \lambda_2 > \lambda_6 > \lambda_8 \tag{46a}$$

approximately

$$\Delta \cong 1 / \lambda_7 \lambda_2$$

$$a_{21} \cong \gamma \rho_1 (\alpha_3 / \alpha^2) e^{-2\alpha \rho_1} \tag{46b}$$

$$a_{22} \cong \gamma / \alpha^2 \tag{46c}$$

taking $b_1 \cong \beta/2$, $\gamma \cong -\beta\omega/u$, equation (42a) gives

$$\Psi_1 = \left[\alpha_3 \left(\frac{\rho_1}{\rho}\right) e^{-2\alpha(\rho_1 - \rho)} + \frac{1}{\rho} \right] \gamma P_2(\cos\vartheta) + f(\rho) \tag{47}$$

If α_3 is not large, the first term in brackets may be neglected anywhere in region II except close to its outer boundary. Then

$$\Psi_1 = -(\beta\omega/u\rho) P_2(\cos\vartheta) + f(\rho) \tag{48}$$

using (6a)

$$B_\varphi = -3\beta \frac{\omega R_0}{u} \cdot \frac{1}{r} \sin\vartheta \cos\vartheta \tag{49}$$

from which, by (3b), the tangent of the "garden-hose angle" is

$$\operatorname{tg} \chi = B_\varphi / B_r = -\omega r \sin\vartheta / u \tag{50}$$

Near the boundary, if $\rho_1, \alpha_3 > 1$, the first term of (47) predominates. Neglecting the second one altogether, equation (6a) gives, on the boundary

$$B_\varphi = -(3\beta\alpha_3\omega/u)\sin\vartheta\cos\vartheta \tag{51}$$

hence, in region III

$$\Psi_1 = -\beta\left(\frac{\omega}{u}\right)\alpha_3\left(\frac{\rho_1}{\rho}\right)^3 P_2(\cos\vartheta) + f_1(\rho) \tag{52}$$

It is evident that Ψ_1 diverges if $\alpha_3 \to \infty$. The reason for this is, roughly, that if all regions are ideal conductors, this model will excite infinite currents by unipolar induction, leading to infinite toroidal fields. If the problem could be solved more realistically, with the flow velocity $\underline{v}$ as one of the unknowns, as α_3 increases without limit the force $[\underline{j} \times \underline{B}]$ on the flow also diverges. The flow then would have been greatly modified near the boundary, and the current density self-limited.

THE ELECTRIC FIELD

The electric field will have both monopole and quadrupole components. We start by evaluating the monopole term $\Psi_{01}(m)$ of Ψ_{01}. From equations (40) and (29a), in region III $\Psi_{01}(m)$ is constant; its value will be chosen as zero since this region extends to infinity. In region II, using (24)

$$\rho\Psi_1(m) = a_{01}e^{2\alpha\rho} + a_{02}$$

from which, by (39)

$$\Psi_{01}(m) = a_{02} + c_2 = \text{constant}$$

and again, this constant must vanish because of continuity with region III. In region I, from (38)

$$\Psi_{01}(m) = \frac{3}{2}(b_1\rho^2 + \beta/\rho)\frac{\omega}{u} + \gamma$$

The quadrupole potential $\Psi_{01}(q)$ gives a nonzero field in all three regions; it may be found by inserting the solution for the quadrupole component of Ψ_1 into equations (38), (39) and (40). In the limit of very high conductivity the quadrupole field in II is also given by

$$\underline{E} = -[\underline{v} \times \underline{B}] = -\underline{i}_\vartheta \; 3\beta\omega R_0 \; \sin\vartheta\cos\vartheta / r$$

showing that Ψ_0 is then independent of r

$$\Psi_0 = -\beta\omega R_0 \; P_2(\cos\vartheta) \quad . \tag{53}$$

In region III Ψ_0 is a harmonic quadrupole potential. From equation (40) one sees that it does not diverge with α_3. It is interesting to note that, in the limiting case, $\underline{E}$ and $\underline{B}$ both vanish in the equatorial plane, so that particles released from the solar equator may freely travel all over it (there is an angle of about 7° between the ecliptic and the plane of solar rotation). While this plane is an equipotential, its potential does not equal the potential at infinity but is

$$\Psi_0 \; (\cos\vartheta = 0) = \frac{1}{2} \left(\beta / R_0\right) \cdot \omega R_0^2 \tag{54}$$

The quantity β/R_0 is of the order of the field intensity at $r = R_0$ and will be taken as one gauss; Ψ_0 then has the order of 10^8 volts. This result may have some connection to the modulation of cosmic-ray intensity by the solar activity cycle, which resembles that produced by an electric field ([Ehmert, 1960] and papers referred to there; most theories of this class have assumed a geocentric electric field). However, the value of Ψ_0 deduced here is too small by at least a factor of 10, and it should be borne in mind that when the solar dipole reverses its direction, as has been observed in 1958 (Babcock, 1959), Ψ_0 is bound to reverse its sign, too.

THE INTERSTELLAR FIELD

To the above rotationally symmetric model one may add a homogeneous "interstellar magnetic field" $\underline{B}_0$, parallel to the symmetry axis. Such a field is represented by a poloidal potential (diverging at infinity)

$$\Psi_2 = \frac{1}{2} B_0 r \cos\vartheta \tag{55}$$

so that equation (29b) will be replaced by

$$\Psi_2 = \sum b_n'' \left(\frac{\rho_1}{\rho}\right)^{n+1} P_n(\cos\vartheta) + \frac{1}{2} B_0 R_0 \rho \cos\vartheta \tag{56}$$

Equations (30) still hold, but the last two of equations (31) have to be modified to

$$\begin{aligned} \frac{1}{2} R_1 B_0 + b_1'' &= b_{11}\, g_{11}(\rho_1) + b_{12}\, g_{12}(\rho_1) \\ \frac{1}{2} R_1 B_0 - 2 b_1'' &= b_{11} \rho_1 g_{11}'(\rho_1) + b_{12} \rho_1 g_{12}'(\rho_1) \end{aligned} \tag{57}$$

The coefficients are solved as before, and it is found (notation of equation 34) that b_{12} is modified by a factor

$$\left(1 - \frac{B_0 R_1}{2\beta} \frac{a}{c}\right) \cong \left(1 + \frac{B_0}{\beta/R_0} \cdot \frac{\rho_1}{2} e^{-2\alpha(\rho_1 - 1)}\right)$$

As has been mentioned, β/R_0 is of the order of the field at $r = R$. If α and ρ_1 are considerably larger than unity, the added term is negligible unless $\underline{B}_0$ is very much larger than β/R_0, which is hardly the case for the interstellar field. Thus the poloidal field in II, and consequently Ψ_1 and Ψ_0, are only negligibly affected by $\underline{B}_0$. The only important term which may undergo a large change is b_1'', which in the limiting case becomes

$$b_1'' = 3\beta/2\rho_1 - RB_0/2 \tag{58}$$

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Figure Captions

1. Division of space into three concentric regions
2. Lines of force of the magnetic field in the limiting case
3. Equipotentials for the electric field in the limiting case

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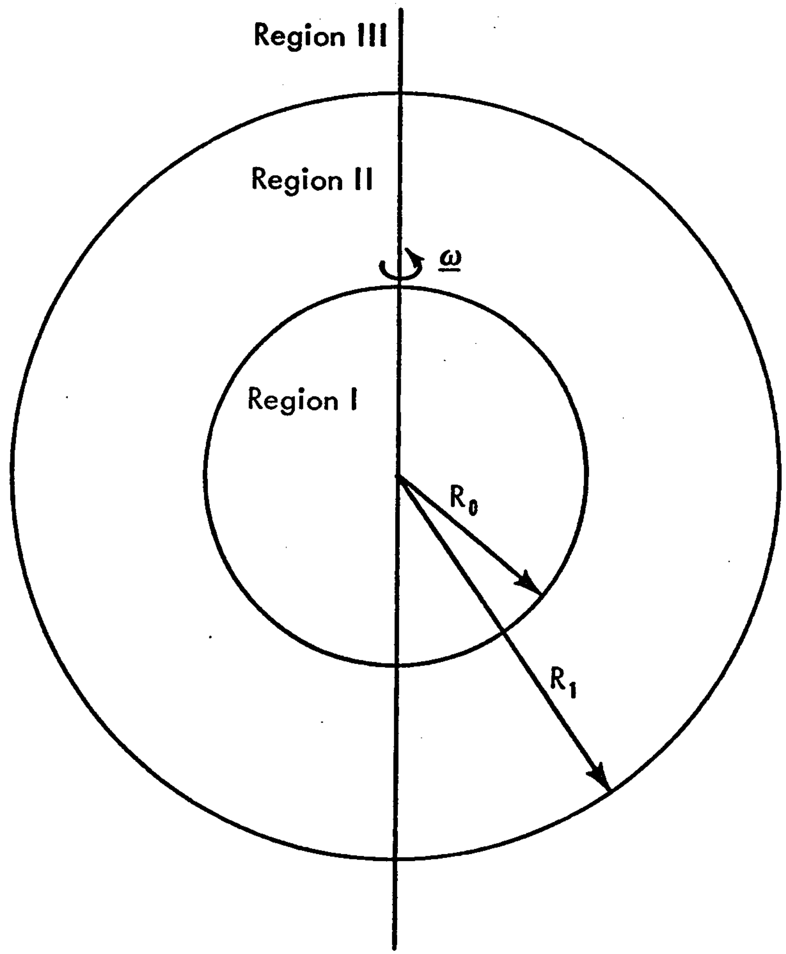

Fig. 1 Division of space into three concentric regions

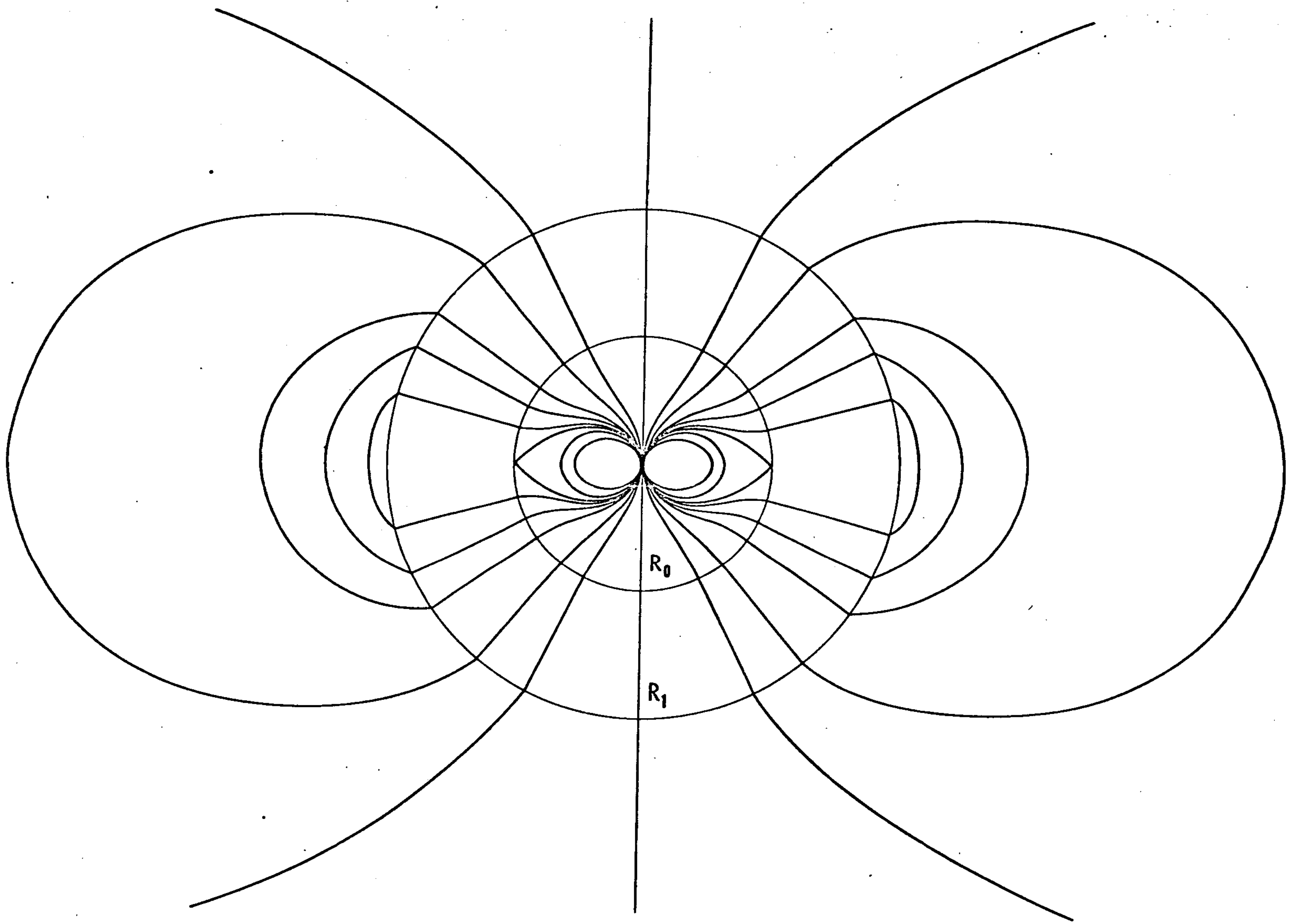

Fig. 2 Lines of Force of the Magnetic Field in the limiting case

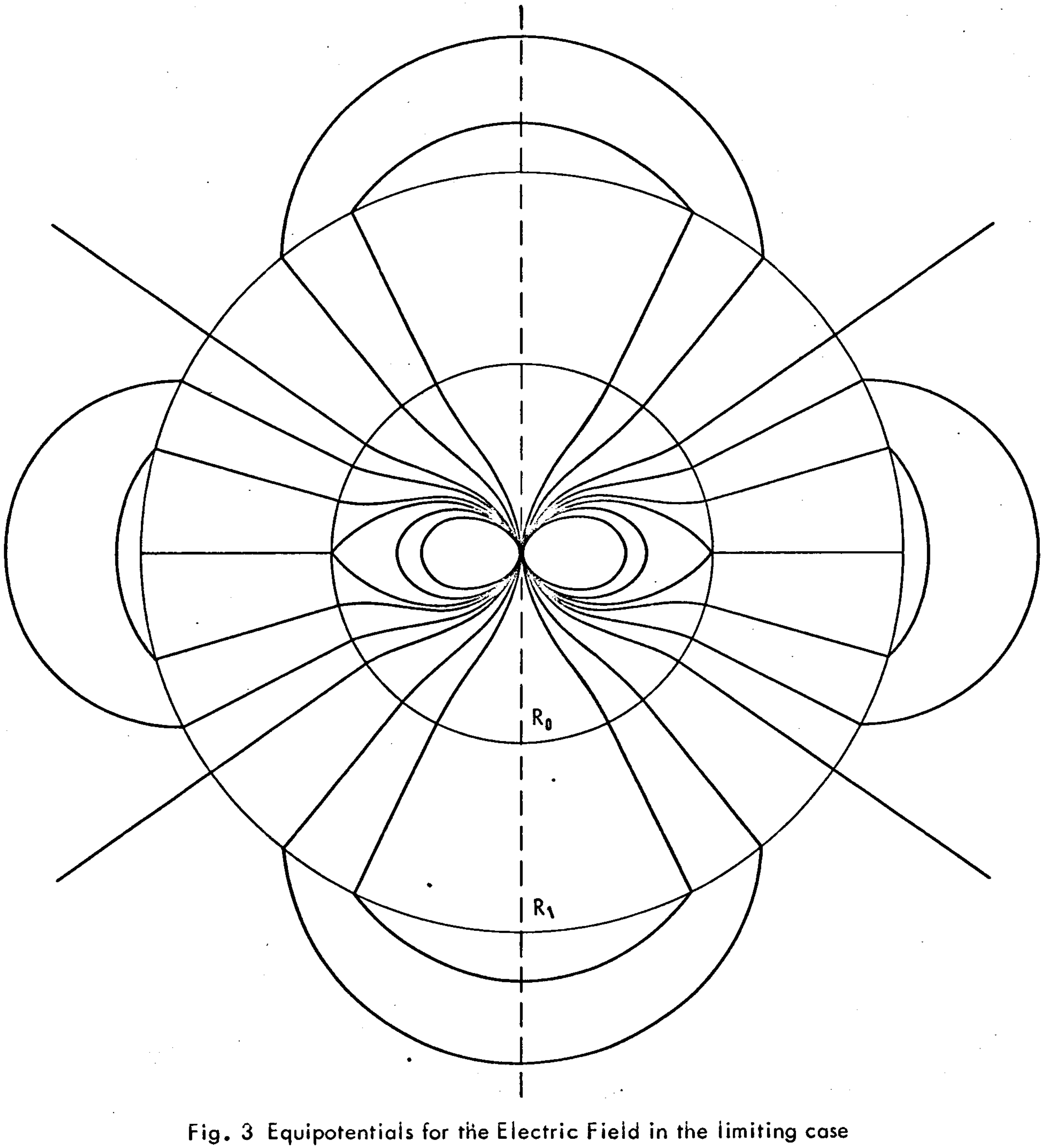

Fig. 3 Equipotentials for the Electric Field in the limiting case